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Best Uniform Polynomial Approximation to Certain Rational Functions*

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1. INTRODUCTION

Let the function f be contained in $C[a, b]$, and let P_n denote the set of polynomials of degree no greater than n, with real coefficients. For every nonnegative integer n there exists a unique polynomial in P_n, p_n^*, such that

$$\max_{a \leqslant x \leqslant b} |f(x) - p(x)| > \max_{a \leqslant x \leqslant b} |f(x) - p_n^*(x)| = E_n(f)$$

for all polynomials p, other than p_n^*, in P_n. We call p_n^* the best uniform polynomial approximation of degree n to f on $[a, b]$. We can characterize p_n^* via the following theorem.

CHEBYSHEV ALTERNATION THEOREM. *Let f be in $C[a, b]$. Let the polynomial p be in P_n, and $\epsilon(x) = f(x) - p(x)$. Then p is the best uniform approximation p_n^* to f on $[a, b]$ if and only if there exist at least $n + 2$ points $x_1, \ldots, x_{n+2}$ in $[a, b]$, $x_i < x_{i+1}$, for which $|\epsilon(x_i)| = \max_{a \leqslant x \leqslant b} |f(x) - p(x)|$, with $\epsilon(x_{i+1}) = -\epsilon(x_i)$.*

Without loss of generality, we can restrict ourselves to the interval $[-1, +1]$.

In this paper we will be concerned with certain functions for which p_n^* can be determined explicitly.

In 1936, Bernstein showed (see Golomb [4]) that if

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),$$

with $a_k \geqslant 0$, and $T_k(x) = \cos k\theta$, $x = \cos \theta$, then

$$p_n^*(x) = \sum_{k=0}^{n} a_k T_k(x)$$

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for all n, if and only if the ratio k_{i+1}/k_i of the indices of two successive nonvanishing coefficients a_{k_i}, $a_{k_{i+1}}$ is an odd integer q_i for each i. The appealing aspect of Bernstein's result is that the best uniform polynomial approximation is merely a truncation of the series giving the function. However, the class of functions of this form is small.

In 1962, Rivlin [9], extending results given by Hornecker [5], considered the class of functions given by

$$f(x) = \sum_{j=0}^{\infty} t^j T_{aj+b}(x) = \frac{T_b(x) - tT_{|b-a|}(x)}{1 + t^2 - 2tT_a(x)}, \tag{1}$$

$a > 0, b \geqslant 0$, a and b integers, $-1 < t < +1$. The best uniform polynomial approximations for $t \neq 0$ are shown to be truncations, with a modification of the last term in the truncated series (cf. Section 3). The results of Bernstein and Rivlin were extended to include rational approximation by Lam and Elliott [8], who consider a series like (1) in which a generalized form of T_k is used.

Rivlin's investigation suggests the possibility of getting similar results by replacing the polynomials T_k in the series expansion by other polynomial sets. In Section 2 we examine the rational functions

$$f(x) = \sum_{j=0}^{\infty} t^j U_{aj+b}(x),$$

where U_k is the Chebyshev polynomial of the second kind of degree k, $U_k(x) = \sin(k+1)\theta/\sin\theta$, $x = \cos\theta$. We find that we must have $a = 2$, and then f in closed form is given by Eq. (2) of Section 2, and p_n^* by

$$\sum_{j=0}^{k} t^j U_{2j+b} - \frac{t^{k+2}}{(1-t)^2(1+t)} U_{2(k-1)+b} - \frac{(t^2 - t - 1)}{(1-t)^2(1+t)} U_{2k+b},$$

for $2k + b \leqslant n < 2(k+1) + b$.

The examination of the error function $\epsilon_n = f - p_n^*$, for f given by $\sum t^j T_{aj+b}$ and $\sum t^j U_{2j+b}$ leads us to consider in Section 3 a general form for ϵ_n, based on the Chebyshev Alternation Theorem. We show that for any $f(x)$ in $C[-1,+1]$, we have $\epsilon_n = \alpha \cos(n\theta + \phi)$, where $x = \cos\theta$, $|\alpha| = E_n(f)$, and the phase angle ϕ is a continuous function of θ, depending on f and n. We then consider a special form of ϵ_n which has ϕ independent of n. In both Rivlin's and our own cases, ϵ_n is of this form. We show that if ϕ is independent of n, for $n = ak$, with a equal to a positive integer and $k = 0, 1, 2,...$, then f is given by (1), with $b = 0$, up to multiplicative and additive constant factors.

2. Approximation by Truncation

Lemma 2.1. *Let* $f(x) = \sum_{j=0}^{\infty} t^j U_{aj+b}(x)$, *with a and b nonnegative integers,* $a > 0$, $-1 < t < +1$. *Then*

$$f(x) = \frac{U_b(x) - tU_{b-a}(x)}{1 + t^2 - 2tT_a(x)}, \tag{2}$$

where we let $U_{-1}(x) = 0$, *and* $U_{b-a}(x) = -U_{a-b-2}(x)$ *for* $a > b + 1$.

Proof. See [10], theorem on page 45.

The value $t = 0$ gives $f(x) = U_b(x)$, which is also the best approximation to f for $n \geqslant b$. This trivial case will be excluded in what follows.

Theorem 2.1. *Let f be given as in Lemma* 2.1. *Let*

$$q(x) = \sum_{j=0}^{k} t^j U_{aj+b}(x) - \delta_k(t)\, U_{a(k-1)+b}(x) - \gamma_k(t)\, U_{ak+b}(x).$$

Then we can solve for $\delta_k(t)$ *and* $\gamma_k(t)$ *such that q is* p_n^* *for* $ak + b \leqslant n < a(k+1) + b$, *exactly for the case* $a = 2$. *When* $a = 2$, *we have for* $k \geqslant 1$

$$\delta_k(t) = \frac{t^{k+2}}{(1-t)^2(1+t)};$$

$$\gamma_k(t) = \frac{(t^2 - t - 1)\, t^{k+1}}{(1-t)^2(1+t)};$$

$$E_n(f) = \frac{2\,|\,t\,|^{k+1}}{(1-t)^2(1+t)}.$$

Proof. The error function $\epsilon(x) = f(x) - q(x)$ can be found in closed form from the difference of the two series. Viewing $\sin(aj + b + 1)\theta$ as $\operatorname{Im}(e^{i(aj+b+1)\theta})$, $\epsilon(x)$ becomes

$$\epsilon(\cos\theta) = \frac{1}{\sin\theta}\operatorname{Im}\Big(t^{k+1}e^{i[a(k+1)+b+1]\theta}\sum_{j=0}^{\infty}(te^{ia\theta})^j + \delta_k(t)\, e^{i[a(k-1)+b+1]\theta} + \gamma_k(t)\, e^{i(ak+b+1)\theta}\Big).$$

Expanding all terms and taking the imaginary parts yields

$$\begin{aligned}\epsilon(\cos\theta) = \frac{1}{\sin\theta}\{&t^{k+2}\sin a\theta\cos[a(k+1)+b+1]\theta/B(\theta)\\ &+ t^{k+1}(1 - t\cos a\theta)\sin[a(k+1)+b+1]\theta/B(\theta)\\ &+ \delta_k(t)\sin[a(k-1)+b+1]\theta + \gamma_k(t)\sin(ak+b+1)\theta\},\end{aligned} \tag{3}$$

where $B(\theta) = 1 + t^2 - 2t \cos a\theta$. The proof depends on the fact that the error can be put into the following form:

$$\epsilon(\cos\theta) = \alpha_k(t) \cos[(ak + b + 1)\theta + \psi]. \tag{4}$$

Equations (3) and (4) can be expanded to give $\cos(ak + b + 1)\theta$ and $\sin(ak + b + 1)\theta$ terms. Since the two expressions for $\epsilon(\cos\theta)$ must be equal, we can equate the coefficients of $\cos(ak + b + 1)\theta$ and $\sin(ak + b + 1)\theta$ to yield the following formal expressions for $\cos\psi$ and $\sin\psi$:

$$\alpha_k(t)\cos\psi = [t^{k+1}\sin a\theta - \delta_k(t)\, B(\theta) \sin a\theta]/\sin\theta B(\theta)$$

and

$$\begin{aligned} -\alpha_k(t)\sin\psi = [&t^{k+1}\cos a\theta - t^{k+2} + \gamma_k(t)\, B(\theta) \\ &+ \delta_k(t)\, B(\theta)\cos a\theta]/\sin\theta B(\theta). \end{aligned}$$

This procedure is valid only if these expressions for $\cos\psi$ and $\sin\psi$ satisfy $\cos^2\psi + \sin^2\psi = 1$ identically in θ. This condition gives upon simplification

$$\begin{aligned} \alpha_k^2(t)\sin^2\theta B(\theta) = {} & t^{2k+2} - 2\gamma_k(t)\, t^{k+2} \\ & + [\delta_k^2(t) + \gamma_k^2(t)]\, B(\theta) + 2\delta_k(t)\,\gamma_k(t)\, B(\theta)\cos a\theta \\ & + 2\delta_k(t)\, t^{k+1}\cos 2a\theta \\ & + 2t^{k+1}\cos a\theta[\gamma_k(t) - t\delta_k(t)]. \end{aligned} \tag{5}$$

Employing several trigonometric identities, Eq. (5) becomes (suppressing the arguments of α_k, δ_k, γ_k)

$$\begin{aligned} & \tfrac{1}{2}\alpha_k^2(1 + t^2) - \alpha_k^2 t\cos a\theta - \tfrac{1}{2}\alpha_k^2(1 + t^2)\cos 2\theta \\ & \quad + \tfrac{1}{2}\alpha_k^2 t\cos(a - 2)\theta + \tfrac{1}{2}\alpha_k^2 t\cos(a + 2)\theta \\ & = [t^{2k+2} - 2\gamma_k t^{k+2} + (\delta_k^2 + \gamma_k^2)(1 + t^2) - 2t\delta_k\gamma_k] \\ & \quad + [-2t(\delta_k^2 + \gamma_k^2) + 2t^{k+1}(\gamma_k - t\delta_k) + 2\delta_k\gamma_k(1 + t^2)]\cos a\theta \\ & \quad + (2\delta_k t^{k+1} - 2t\delta_k\gamma_k)\cos 2a\theta, \end{aligned} \tag{6}$$

and must hold identically in θ. The various arguments of cosine are 0θ, $a\theta$, $2a\theta$, 2θ, $(a - 2)\theta$, $(a + 2)\theta$. If any one of the last three arguments is not actually equal to another argument, then its coefficient must be identically zero, which means $t = 0$ or $\alpha_k = 0$. But $\alpha_k = 0$ implies f is a polynomial, which can only occur in the trivial case $t = 0$.

We check for values of a which permit solutions in the nontrivial case and find only $a = 2$. When $a = 2$, equating coefficients of cosine of 0θ, 2θ, 4θ in (6) yields the three equations

$$\tfrac{1}{2}\alpha_k^2(1 + t + t^2) = t^{2k+2} - 2\gamma_k t^{k+2} - 2t\delta_k\gamma_k + (\delta_k^2 + \gamma_k^2)(1 + t^2), \tag{7}$$

$$\tfrac{1}{2}\alpha_k^2(1 + t)^2 = 2t(\delta_k^2 + \gamma_k^2) - 2(1 + t^2)\,\delta_k\gamma_k - 2t^{k+1}(\gamma_k - t\delta_k), \tag{8}$$

$$\tfrac{1}{2}\alpha_k^2 t = -2t\delta_k\gamma_k + 2t^{k+1}\delta_k. \tag{9}$$

The value of α_k^2 obtained from (9) is next substituted in (8) and (7). We get two equations from which we can eliminate $(\delta_k + \gamma_k)^2$ to yield $\delta_k + t\gamma_k = -t^{k+3}/(1 - t^2)$. Writing γ_k in terms of δ_k, we find $\delta_k(t) = t^{k+2}/(1 - t)^2(1 + t)$, and then

$$\gamma_k(t) = (t^2 - t - 1)t^{k+1}/(1 - t)^2(1 + t)$$

and

$$\alpha_k(t) = 2t^{k+1}/(1 - t)^2(1 + t).$$

Evaluating $\cos\psi$ and $\sin\psi$ gives

$$\cos\psi = \frac{(1 - 2t - t^2)\sin 2\theta + t^2 \sin 4\theta}{2\sin\theta(1 + t^2 - 2t\cos 2\theta)}, \tag{10a}$$

$$\sin\psi = \frac{(1 + 2t) - (1 + t)^2\cos 2\theta + t^2\cos 4\theta}{2\sin\theta(1 + t^2 - 2t\cos 2\theta)}. \tag{10b}$$

It is possible to analyze Eqs. (10a,b) to show directly that ψ goes from 0 to π as θ varies from 0 to π. However, it is far easier to change the expansion of (4) so that $\epsilon(\cos\theta) = \alpha_k(t)\cos[(ak + b)\theta + (\theta + \psi)]$. Letting $\phi = \theta + \psi$, $\cos\phi$ and $\sin\phi$ are given by

$$\sin\phi = \frac{(1 - t^2)\sin a\theta}{1 + t^2 - 2t\cos a\theta}, \tag{11a}$$

$$\cos\phi = \frac{-2t + (1 + t^2)\cos a\theta}{1 + t^2 - 2t\cos a\theta}. \tag{11b}$$

These functions appear in Rivlin [9], and it is known that as θ increases from 0 to π, ϕ increases continuously from 0 to $a\pi$. The argument $(2k + b)\theta + \phi$ increases continuously from 0 to $[2(k + 1) + b]\pi$ as θ increases from 0 to π, so the error takes its extreme values $\pm|\alpha_k(t)|$ with alternating sign at the $2(k + 1) + b + 1$ points in $0 \leqslant \theta \leqslant \pi$ at which $\cos[(2k + b)\theta + \phi]$ will be ± 1. Invoking the Chebyshev Alternation Theorem, $q(x)$ is $p_n^*(x)$, with $E_n(f) = |\alpha_k(t)|$, for $2k + b \leqslant n < 2(k + 1) + b$, $k \geqslant 1$. Q.E.D.

It is interesting to note that if we allow even greater variation of the two modifying terms, as by taking

$$q(x) = \sum_{j=0}^{k} t^j U_{aj+b}(x) - \delta_k(t)\, U_{a(k-\sigma)+b}(x) - \gamma_k(t)\, U_{a(k-\rho)+b}(x),$$

$0 \leqslant \sigma < \rho \leqslant k$, then no set of values other than $\sigma = 1$, $\rho = 0$, $a = 2$ admits a nontrivial solution.

Rename the function given in (2) as $f(U, a, b)$ and the function given in (1) as $f(T, a, b)$. One can show that $f(U, 2, b)$ differs from $f(T, a, b)$ in the following sense. For $b_1 > 2$ there are no values of a, b_2, α, and β such that $f(U, 2, b_1) = \alpha f(T, a, b_2) + \beta$ for all values of t.

The Chebyshev Alternation Theorem implies the following lemma.

LEMMA 2.2. *If p_n^* is the best uniform approximation in P_n to $f \in C[a, b]$, then for arbitrary real numbers α and β, $\alpha p_n^* + \beta$ is the best uniform approximation in P_n to $\alpha f + \beta$ on $[a, b]$, and $E_n(\alpha f + \beta) = |\alpha| E_n(f)$.*

This gives an obvious extension to Theorem 2.1, analogous to the corollary in [9].

3. A UNIQUENESS RESULT

According to the Chebyshev Alternation Theorem, $f(x) - p_n^*(x)$ takes its extreme values $\pm\alpha$, $|\alpha| = E_n(f)$, with alternating signs at least $n + 2$ times in $[a, b]$. When $[a, b] = [-1, +1]$, this corresponds to extrema of $\epsilon_n = f(\cos\theta) - p_n^*(\cos\theta)$ in $0 \leqslant \theta \leqslant \pi$. *The error ϵ can always be described as* $\alpha\cos(n\theta + \phi)$. The function ϕ gives the phase of the error. That is, it describes the behavior of $f - p_n^*$ as a variation in the argument of the cosine function. The choice of ϕ depends on f and n. By choosing $\phi(0)$ to be in $[0, 2\pi)$, and considering $e^{i\phi(\theta)}$ to be on the Riemann surface corresponding to e^z, we see that $\phi(\theta)$ varies continuously as θ goes from 0 to π. The argument $n\theta + \phi$ will have a range including the closed interval with endpoints at $\phi(0)$ and $n\pi + \phi(\pi)$. The range must provide as many extrema of cosine as required for $f - p_n^*$.

For the functions $f(T, a, b)$ and $f(U, 2, b)$ we have $\epsilon_n = \alpha_k \cos[(ak+b)\theta + \phi]$, for $ak + b \leqslant n < a(k + 1) + b$, in both cases. Here we see that ϕ is a continuous function of θ, but it is independent of the choice of k. In fact, we may include the $b\theta$ term in $\phi' = b\theta + \phi$, and ϕ' is still independent of n. We shall determine all functions whose error of approximation contains such a "constant phase" for all $n \geqslant 0$. The following lemmas are required.

LEMMA 3.1. *Let $f \in C[-1, +1]$ be such that p_{ak}^* satisfies $f(\cos\theta) - p_{ak}^*(\cos\theta) = \alpha_k \cos(ak\theta + \phi)$ for $k \geqslant 0$, where a is a positive integer and ϕ is a continuous function of θ, independent of k. Let α_1 and α_2 be nonzero. Let* $\cos\phi$ *and* $\sin\phi$ *be even and odd functions of θ, respectively. Then f is a rational function, and the degree of the numerator is no greater than the degree of the denominator.*

Proof. Writing $u(\theta) = \cos\phi$, $v(\theta) = \sin\phi$, we can express these functions as formal Fourier series,

$$u(\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta \qquad \text{and} \qquad v(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta.$$

Since

$$p_{ak}^*(\cos\theta) = f(\cos\theta) - \alpha_k \cos ak\theta u(\theta) + \alpha_k \sin ak\theta v(\theta)$$

and $k = 0$ gives

$$f(\cos\theta) = \alpha_0 u(\theta) + p_0^*, \tag{12}$$

we can express p_{ak}^*, for $k \geqslant 1$, as a Fourier cosine series whose coefficients involve only the a_n's and b_n's. The coefficients of $\cos n\theta$ for $n > ak$ must be zero in this series. It follows that

$$2\alpha_0 a_n - \alpha_k(a_{n+ak} + a_{n-ak} - b_{n+ak} + b_{n-ak}) = 0 \tag{13}$$

for $n > ak$ (for each fixed $k \geqslant 1$). Multiplying (13) by x^n and summing for $n \geqslant ak + 1$ (for k fixed), we have

$$\begin{aligned}[2\alpha_0 x^{ak} - \alpha_k(1 + x^{2ak})]\, U(x) + \alpha_k(1 - x^{2ak})\, V(x) \\ = 2\alpha_0 x^{ak} \sum_{n=0}^{ak} a_n x^n - \alpha_k \sum_{n=0}^{2ak} a_n x^n - \alpha_k a_0 x^{2ak} + \alpha_k \sum_{n=0}^{2ak} b_n x^n - \alpha_k b_0 x^{2ak},\end{aligned} \tag{14}$$

where

$$U(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad V(x) = \sum_{n=1}^{\infty} b_n x^n.$$

Here we may note that the coefficients of U and V in (14) are the characteristic polynomials of the parts of the linear recurrence relation (13) involving only the a_n's and b_n's, respectively. Denote these characteristic polynomials by $u_k(x)$ and $v_k(x)$. Rewrite (14) as

$$u_k(x)\, U(x) + v_k(x)\, V(x) = r_k(x), \tag{15}$$

and we see that when $\alpha_k \neq 0$, the degrees of u_k and v_k equal $2ak$, and the degree of r_k is no greater than $2ak$. Take (15) for $k = 1$ and multiply the equation by $\alpha_2(x^{2a} + 1)$. From this we subtract (15) taken for $k = 2$ and multiplied by α_1. The $V(x)$ term is eliminated, and we have

$$[\alpha_2(x^{2a} + 1)\, u_1(x) - \alpha_1 u_2(x)]\, U(x) = \alpha_2(x^{2a} + 1)\, r_1(x) - \alpha_1 r_2(x). \tag{16}$$

This tells us $U(x)$ is rational, and consequently, so is $V(x)$. Equation (16) also gives bounds on the degrees of the numerator and denominator of $U(x)$. The coefficient of $U(x)$ simplifies to $-2\alpha_2 x^a(\alpha_0 x^{2a} - \alpha_1 x^a + \alpha_0)$. The right-hand side of (16) contains x^a as a factor, as well. After cancellation of x^a, we find $2a$ as the upper bound for the degree of the denominator of $U(x)$. However, if in (15) we write $U(x) = p(x)/q(x)$ and $V(x) = s(x)/t(x)$ and then clear fractions, it immediately follows that the degree of $p(x)$ is not greater than the degree of $q(x)$.

Noting that $u(\theta) = \operatorname{Re} U(e^{i\theta})$ and $v(\theta) = \operatorname{Im} V(e^{i\theta})$, and letting

$$p(x) = \sum_{j=0}^{\sigma} p_j x^j, \qquad q(x) = \sum_{j=0}^{\tau} q_j x^j, \qquad \sigma \leqslant \tau,$$

then

$$u(\theta) = \frac{(\sum_{j=0}^{\sigma} p_j \cos j\theta)(\sum_{k=0}^{\tau} q_k \cos k\theta) + (\sum_{j=1}^{\sigma} p_j \sin j\theta)(\sum_{k=1}^{\tau} q_k \sin k\theta)}{(\sum_{k=0}^{\tau} q_k \cos k\theta)^2 + (\sum_{k=1}^{\tau} q_k \sin k\theta)^2}.$$

Multiplying the summations and using

$$\cos j\theta \cos k\theta = \tfrac{1}{2}\cos(j-k)\theta + \tfrac{1}{2}\cos(j+k)\theta,$$

and

$$\sin j\theta \sin k\theta = \tfrac{1}{2}\cos(j-k)\theta - \tfrac{1}{2}\cos(j+k)\theta, \tag{17}$$

the highest order terms in the last form of $u(\theta)$ are $2q_0q_\tau \cos \tau\theta$ in the denominator, and $p_0 q_\tau \cos \tau\theta$ (and $q_0 p_\sigma \cos \sigma\theta$ if $\sigma = \tau$) in the numerator.

Since f is given by (12), we note that some terms in its numerator may cancel, and the lemma is proved. Q.E.D.

Although the proof requires α_1 and α_2 to be nonzero in order to get a meaningful expression for (16), if f is a constant, then $\alpha_1 = \alpha_2 = 0$, but f still satisfies the statement of the lemma.

LEMMA 3.2. *Let p_a be a polynomial of degree a. If*

$$(1 + \alpha^2) - 2\alpha \cos a\theta - p_a^2(\cos\theta) = K(1 - \cos^2 a\theta) \tag{18}$$

for some constant $K \neq 0$, then $p_a(\cos\theta)$ must be of the form $\rho_0 + \rho_a \cos a\theta$.

Proof. Let

$$p_a(\cos\theta) = \sum_{j=0}^{a} \rho_j \cos j\theta.$$

The function $p_a^2(\cos\theta)$ is taken as a trigonometric polynomial by using (17), and (18) becomes

$$(1+\alpha^2) - \frac{1}{2}\left(\rho_0^2 + \sum_{j=0}^{a} \rho_j^2\right) - \sum_{r=1}^{a-1} \frac{1}{2}\left(\sum_{j=0}^{r} \rho_j\rho_{r-j} + 2\sum_{j=0}^{a-r} \rho_j\rho_{j+r}\right)\cos r\theta$$
$$- \left(2\alpha + \frac{1}{2}\sum_{j=0}^{a} \rho_j\rho_{a-j} + \rho_0\rho_a\right)\cos a\theta - \frac{1}{2}\sum_{r=a+1}^{2a}\left(\sum_{j=r-a}^{a} \rho_j\rho_{r-j}\right)\cos r\theta$$
$$= \tfrac{1}{2}K(1 - \cos 2a\theta).$$

Equating the coefficients of $\cos 2a\theta$ yields $K = \rho_a^2$, and then the $\cos(2a - 1)\theta$ term gives $\rho_{a-1}\rho_a = 0$, or $\rho_{a-1} = 0$. Assuming $\rho_{a-j} = 0$ for $j = 1, 2,..., \omega - 1$, then the $\cos(2a - \omega)\theta$ term gives $\rho_{a-\omega} = 0$. By induction, it follows that $\rho_{a-j} = 0$ for $j = 1, 2,..., a - 1$. Finally, the constant terms give $\rho_0^2 = (1 + \alpha^2) - K$. Q.E.D.

THEOREM 3.1. *Let f be in $C[-1, +1]$, and let a be a positive integer. Assume $f(\cos\theta) - p_{ak}^*(\cos\theta) = \alpha_k \cos(ak\theta + \phi)$ for $k \geqslant 0$, where ϕ is a continuous function of θ, independent of k. Assume α_1 and α_2 are nonzero. Then f is a rational function of the form $f(T, a, 0)$, up to multiplicative and additive constant factors.*

Remarks. If we were to assume that the phase angle ϕ satisfies $\phi(0) = m\pi$, $\phi(\pi) = (a + m)\pi$ for some integer m, then we are assured of a sufficient number of alternations of the error to have $p_{ak}^* = p_n^*$ for $ak \leqslant n < a(k + 1)$. This also forces the error of approximation to have extrema at both endpoints of the interval. The error form is assumed for $k \geqslant 0$ in order to start with p_0^*. Lemma 2.2 tells us that if f has this form for the error of approximation, then so does $\alpha f + \beta$, for any real α and β.

Proof. We first derive a formula for $f(\cos\theta)$ in terms of $p_a^*(\cos\theta)$ and trigonometric functions. Letting $k = 0$, we have $f(\cos\theta) - p_0^* = \alpha_0 \cos\phi$. By Lemma 2.2, we can let $\alpha_0 = 1$ and $p_0^* = 0$ since choosing particular values of α_0 and p_0^* is equivalent to modifying f by multiplicative and additive constant factors. Now taking $k = 1$, we have

$$f(\cos\theta) - p_a^*(\cos\theta) = \alpha_1(\cos a\theta \cos\phi - \sin a\theta \sin\phi). \tag{19}$$

Substituting $\cos\phi = f(\cos\theta)$ and solving (19) for $\sin\phi$ yields

$$\sin\phi = \frac{(\alpha_1 \cos a\theta - 1) f(\cos\theta) + p_a^*(\cos\theta)}{\alpha_1 \sin a\theta}.$$

Calculating $\cos^2\phi + \sin^2\phi = 1$ gives a quadratic equation in $f(\cos\theta)$. Solving for f and simplifying gives

$$f(\cos\theta) = \frac{(1 - \alpha_1 \cos a\theta)\, p_a^*(\cos\theta) \pm [g_a(\cos\theta)]^{1/2}}{(1 + \alpha_1^2) - 2\alpha_1 \cos a\theta}, \tag{20}$$

where $g_a(\cos\theta) = \alpha_1^2 \sin^2 a\theta[(1 + \alpha_1^2) - 2\alpha_1 \cos a\theta - p_a^{*2}(\cos\theta)]$. From Eq. (20) it follows that f is an even function in θ, implying that as functions of θ, $\cos\phi$ is even and $\sin\phi$ is odd. By Lemma 3.1, $f(\cos\theta)$ is a rational function.

Next we determine p_a^* explicitly. Since f is rational, the function $g_a(\cos\theta)$ must be the perfect square of a polynomial with real coefficients. The term

$\alpha_1^2 \sin^2 a\theta = \alpha_1^2(1 - \cos^2 a\theta) = \frac{1}{2}\alpha_1^2(1 - \cos 2a\theta)$ has only simple zeros, so it has no perfect square factors. Therefore, $(1 - \cos^2 a\theta)$ must be a factor of

$$(1 + \alpha_1^2) - 2\alpha_1 \cos a\theta - p_a^{*2}(\cos\theta). \tag{21}$$

Since these two terms have the same degree, they can differ only by a multiplicative constant. Applying Lemma 3.2, we have $p_a^*(\cos\theta) = \mu + \nu\cos a\theta$, with $\mu^2 + \nu^2 = 1 + \alpha_1^2$. The coefficient of $\cos a\theta$ must be zero in (21), so $\alpha_1 = -\mu\nu$. It follows that $(\mu - \nu)^2 = (1 + \alpha_1)^2$ and $(\mu + \nu)^2 = (1 - \alpha_1)^2$. Therefore,

$$\mu - \nu = \pm(1 + \alpha_1) \tag{22a}$$

and

$$\mu + \nu = \pm(1 - \alpha_1). \tag{22b}$$

The four sets of equations arising from (22a,b) lead to the following two solutions of $p_a^*(\cos\theta)$ and their negatives:

$$1 - \alpha_1 \cos a\theta, \tag{23a}$$

$$-\alpha_1 + \cos a\theta. \tag{23b}$$

We now substitute for p_a^* and g_a in Eq. (20) to solve for $f(\cos\theta)$. In view of Lemma 2.2, we need consider only the four solutions of f arising from (23a,b) and can neglect the negatives of these f's, which arise from the other choices of p_a^*. Equation (23a) gives

$$f(\cos\theta) = +1 \tag{24a}$$

and

$$f(\cos\theta) = \frac{1 - 2\alpha_1\cos a\theta + \alpha_1^2\cos 2a\theta}{(1 + \alpha_1^2) - 2\alpha_1\cos a\theta}, \tag{24b}$$

while (23b) gives

$$f(\cos\theta) = \frac{-2\alpha_1 + (1 + \alpha_1^2)\cos a\theta}{(1 + \alpha_1^2) - 2\alpha_1\cos a\theta} \tag{24c}$$

and

$$f(\cos\theta) = +\cos a\theta. \tag{24d}$$

Equations (24a,d) are not admissible solutions since they do not have α_1 and α_2 both nonzero, and were excluded initially.

By Lemma 3.1, we know that the degree of the numerator is not greater than the degree of the denominator of f, so (24b) is not an admissible solution. We need now show that (24c) satisfies the statement of the theorem.

Rivlin showed that $f(T, a, 0) - p_{ak}^* = (t^{k+1}/(1-t^2))\cos(ak\theta + \phi)$, where

$$p_n^*(x) = \sum_{j=0}^{k} t^j T_{aj}(x) + \frac{t^{k+2}}{1-t^2} T_{ak}(x), \qquad \text{for} \quad ak \leqslant n < a(k+1), \quad k \geqslant 0,$$

and ϕ is given by (11). This gives us the desired result for $f(T, a, 0)$.

We may note that our assumption on the error form implies that if $E_{ak}(f) \neq 0$, then $E_{a(k+1)}(f) < E_{ak}(f)$, because the approximation will change for $k+1$. Since $E_{ak}(f) = |\alpha_k|$ and $\alpha_0 = 1$, it follows that $-1 < \alpha_1 < +1$. Then $f(\cos\theta)$ in (24c) is the same as $\cos\phi$ in Eq. (11b). It is easy to check that $\cos\phi = \alpha f(T, a, 0) + \beta$, where $\alpha = (1-t^2)/t$ and $\beta = -1/t$. From Rivlin's result and Lemma 2.2, we know that $\alpha f(T, a, 0) + \beta$ for any real α and β will have an error of approximation of the form $\alpha_k \cos(ak\theta + \phi)$ for $k \geqslant 0$, with ϕ independent of k. So $f(T, a, 0)$ is the only function which satisfies the theorem, up to the choice of α and β. Q.E.D.

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